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Categorical relationships between Goguen sets and "two-sided" categorical models of linear logic

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Abstract

The relationships between "two-sided" categorical models of linear logic and Goguen sets is investigated. In particular, we show that only certain Goguen sets can be represented as Chu spaces, while it is possible to represent any Goguen set as a Dialectica space. In addition, we discuss the benefits of these representations.
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1. Introduction

Category theory [10] is a very high-level abstract mathematical theory that unifies all branches of mathematics. Category theorists define collections of *objects* and *morphisms* or *arrows* (i.e., maps) between objects that are called *categories*. In addition, they investigate the internal structure of a category and/or the relationships between categories. Goguen, using the language of category theory, defined *Goguen* sets and their categories in [7]. Goguen sets are the categorical equivalent of all possible forms of fuzzy sets. Topoi are special categories that have been proposed as a possible foundation of all mathematics and computer science, so it would be interesting if Goguen categories were topoi. However, Goguen categories lack certain properties that would classify them as topoi. To remedy this deficiency, Barr in [4] has given a general framework in which one can redefine Goguen sets so that their category forms a topos.

In 1958, Kurt Gödel published in the journal *Dialectica* an interpretation of intuitionistic arithmetic in a quantifier-free theory of functionals of finite type, which has come to be known as *Dialectica Interpretation* [2]. de Paiva [4] presents her own categorical version of the Dialectica interpretation.

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In her thesis, she presents two categories with morphisms that correspond to the Dialectica inter-
pretation of implication. The work was expanded in [12], where she presented one more categorical
version of the Dialectica interpretation. The term *Dialectica* space has been used in [12] as a gen-
eral name of an object of any *Dialectica* category. The Dialectica categories are models of linear
logic [5]. This is particularly interesting, as categorical semantics model derivations (i.e., proofs)
and are not just used to deduce whether a theorem is true or not. Another widely known categorical
model of linear logic is based on the Chu construct described in [3]. Chu categories are built by
means of the Chu construct, that is the enrichment of the category **Chu**(V, k) over the category V.
Chu spaces are objects of a category **Chu**$(\mathbf{Set}, C)$, where **Set** is the category of sets and functions
between them and C is a set. Note that any category **Chu**$(\mathbf{Set}, C)$ is usually denoted by **Chu**(C). Chu
spaces and Dialectica spaces are called "two-sided" models of linear logic because they consist of a
co-variant and a contra-variant component. This way, linear negation can be obtained by exchanging
the rôles of the two.

Linear logic and fuzzy sets are particularly important to computer scientists, so the discovery of
any link between the two theories would be of great interest. The motivation behind the present
work is to show that such a link exists. One way to do this is by representing Goguen sets as either
Chu spaces or Dialectica spaces. We show that it is possible to represent only some Goguen sets as
Chu spaces, but, surprisingly, it is possible to represent any Goguen set as a Dialectica space.

In what follows we will make use of Zadeh's image and preimage operators. In particular, given
a lattice L and a function $f : X \to Y$, the image $f^{\to} : L^X \to L^Y$ and the preimage $f^{\leftarrow} : L^X \to L^Y$ are
defined by

$$f^{\to}(a)(y) = \bigvee \{a(x) : x \in X, f(x) = y\}$$

and

$$f^{\leftarrow}(b) = b \circ f,$$

respectively. In addition, it is a fact that these two operators form an adjunction $f^{\to} \dashv f^{\leftarrow}$, so for
all $a \in L^X$ and all $b \in L^Y$ it holds that $a \leqslant f^{\leftarrow}(f^{\to}(a))$ and $f^{\to}(f^{\leftarrow}(b)) \leqslant b$.

2. Goguen sets

In order to introduce the concept of a Goguen set, Goguen himself used the notion of a frame or
locale, that is a complete lattice where binary meets distribute over joins:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}.$$

This amounts to saying that a lattice A is a frame if and only it is a complete Heyting algebra. This
can be proved by defining the operator $\to$ in a frame as follows:

$$a \to b = \bigvee \{c : c \wedge a \leqslant b\}.$$

For the remainder of this note, L is a frame. We now give the definition of Goguen sets.

Definition 2.1. Let L be a frame. A Goguen set is a pair (S, σ), where S is a set and $\sigma \in L^S$. Given two Goguen sets (S, σ) and (T, τ) a map $f : (S, \sigma) \to (T, \tau)$ is a function $f : S \to T$ such that $\sigma \leqslant f^{\leftarrow}(\tau)$.

Goguen sets are also known as L-fuzzy sets. Following Goguen, we define the category $\mathbf{SET}(L)$ with Goguen sets as objects and maps between Goguen sets as morphisms.

Goguen sets are general enough. So we will not consider any other form of the concept of fuzzy set, such as "intuitionistic" fuzzy sets [1]. [1] The main reason is that the L-intuitionistic fuzzy subsets of a set have been shown in [8] to be order-isomorphic to the L^*-fuzzy subsets of that same set, where

$$L^* = \{(a, b) \in L \times L : a \leqslant b'\}$$

and L^*-fuzzy subsets are understood in this case in the usual sense of the L-fuzzy subsets as in [6].

Barr in [4] notes that the categories $\mathbf{SET}(L)$ are not topoi. Thus he proposes an extension to the concept of Goguen sets so that the resulting categories are topoi. The main drawback to this extension is that fuzzy sets are not fuzzy enough. In particular, two members of a fuzzy set are either equal or are not equal, which is just the "crisp" definition. Following Barr, we define a Goguen set to be a triplet (S, σ, η), where $\eta : S \times S \to L$ is a measure of the degree to which two members of the Goguen set S are equal. After this modification, the resulting categories are topoi. Here is one possible definition of equality between members:

$$\eta(s_1, s_2) = \begin{cases} 0 & \text{if } (s_1 \neq s_2) \wedge [\sigma(s_1) = \sigma(s_2)], \\ \min\{\sigma(s_1), \sigma(s_2)\} & \text{if } (s_1 = s_2) \wedge [\sigma(s_1) \neq \sigma(s_2)], \\ 1 & \text{otherwise}, \end{cases}$$

where $\wedge$ is the standard Boolean conjunction operator.

3. From Goguen sets to Chu and Dialectica spaces

A *Chu space* over an alphabet Σ (i.e., an arbitrary set whose structure is of no importance) is a triplet (X, r, A), where X and A are arbitrary sets and $r : X \times A \to \Sigma$ is a function. Function r relates the elements of X with the elements of A. For example, suppose that $\Sigma = \{0, 1\}$ and that A stands for the set of open subsets of X. Then, $r(x, a) = 1$ if x belongs to the open subset a, else $r(x, a) = 0$. Following a similar way of thinking, one can represent any relational structure (e.g., groups, vector spaces, categories, etc.) as a Chu space.

Let $\mathscr{A} = (X, r, A)$ and $\mathscr{B} = (Y, s, B)$ be two Chu spaces. Then a transformation from $\mathscr{A}$ to $\mathscr{B}$ is just a pair of functions $(f, \bar{f})$, $f : X \to Y$, $\bar{f} : B \to A$, such that

$$s(f(x), b) = r(x, \bar{f}(b)), \quad \forall x \in X, \forall b \in B, \tag{1}$$

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[1] The term "intuitionistic" is a misnomer. It has been used because both in "intuitionistic" fuzzy set theory and in intuitionistic logic there is no complementarity between a proposition and its negation. In our opinion, "intuitionistic" fuzzy sets should be better called *non-symmetric* fuzzy sets.

or in displayed form

$$
\begin{array}{ccc}
X \times B & \xrightarrow{\ f \times \mathrm{id}_B\ } & Y \times B \\
{\scriptstyle \mathrm{id}_X \times \tilde{f}} \downarrow & & \downarrow {\scriptstyle s} \\
X \times A & \xrightarrow[\ r\ ]{} & \Sigma
\end{array}
\tag{2}
$$

This condition is called the *adjointness condition*. We build a category **Chu**(Σ) with objects all Chu spaces and with morphisms pairs of functions that fulfill the adjointness condition. Morphism composition is the usual composition of functions pairwise. For any Chu space (X, r, A) it is easy to verify that the identity morphism is the pair of functions $(\mathrm{id}_X, \mathrm{id}_A)$.

Let L be a frame. Then we consider a subcategory of **SET**(L) with the following property: for each pair of objects (S, σ) and (T, τ), a function $f : S \to T$ is a morphism between them iff $\sigma = f^{\leftarrow}(\tau)$. We call this subcategory **SET**$(L)_=$. Although the restriction imposed on the morphisms of **SET**$(L)_=$ may seem too strong, the following proposition shows that there are enough morphisms in **SET**$(L)_=$.

Proposition 3.1. *Let (S, σ) and (T, τ) be Goguen sets and $f : S \to T$ an injective function satisfying $f^{\to}(\sigma) = \tau$. Then $f : (S, \sigma) \to (T, \tau)$ is a morphism such that $\sigma = f^{\leftarrow}(\tau)$.*

This proposition is immediate from the following remarks:

Remark 3.1. *If $f : L \to M$, $g : L \leftarrow M$ are isotone maps between posets, then $f \dashv g$ implies $f \circ g \circ f = f$. If it is also assumed that f is injective, then $g \circ f = \mathrm{id}_L$.*

Remark 3.2. *If $f : X \to Y$ is injective, then $f^{\to}$ is injective, which implies that $f^{\leftarrow} \circ f^{\to} = \mathrm{id}_{L^X}$.*

Now we proceed with the definition of the functor $\mathfrak{F}$ from the subcategory **SET**$(L)_=$, for some fixed L, to the category **Chu**(L). We first define the object part:

Definition 3.1 (Object part). Let (S, σ) be an object of **SET**$(L)_=$. Then functor $\mathfrak{F}$ maps it to the Chu space $(S, r, \{\sigma\})$, where $r(s, \sigma) = \sigma(s)$.

The following result is direct consequence of the previous definition:

Corollary 3.1. *Functor $\mathfrak{F}$ is injective on objects.*

We now define the morphism part of the functor:

Definition 3.2 (Arrow part). Suppose that $\mathfrak{F}(S, \sigma) = (S, r, \{\sigma\})$ and $\mathfrak{F}(T, \tau) = (T, s, \{\tau\})$. Moreover, suppose that $f : (S, \sigma) \to (T, \tau)$ is an arrow of **SET**$(L)_=$. Then $\mathfrak{F}(f) = (f, g)$, where $g(\tau) = \sigma$.

The following result is a direct consequence of the above definitions:

Theorem 3.1. *Any subcategory* $\mathbf{SET}(L)_=$ *fully embeds*[2] *into a category* $\mathbf{Chu}(L)$.

Goguen sets can be represented more naturally as Dialectica spaces. Generally speaking, Dialectica spaces are Chu spaces with more morphisms between them. Although there at least three different families of Dialectica categories, here we are interested only in the categories $\mathbf{Dial}_L\mathbf{Set}$, where L is a lineale that is, a monoidal poset with additional structure. Here are the relevant definitions borrowed from [13].

Definition 3.3. A monoidal poset is a *poset* $(L, \leqslant)$ with a given symmetric monoidal structure $(L, \circ, 1)$. That is, a set L equipped with a binary relation $\leqslant$, together with a monoid structure $(\circ, 1)$ consisting of a (order-preserving) multiplication $\circ : L \times L \to L$ and a distinguished object 1 of L. We write a monoidal poset as a quadruple $(L, \leqslant, \circ, 1)$.

Operator '$\circ$' is a logical conjunction operator, which is not necessarily idempotent. In addition, 1 is not necessarily the top element of L. Suppose now that L is a monoidal poset and a, b are elements of L. If there is an $x \in L$, which is the largest element of L such that $a \circ x \leqslant b$, then this element is denoted $a \multimap b$.

Definition 3.4. A *lineale* (or close poset) is a monoidal poset such that $a \multimap b$ exists for all a and b in L. We write a lineale as a quintuple $(L, \leqslant, \circ, 1, \multimap)$.

In addition, it holds that $z \circ x \leqslant y$ iff $z \leqslant (x \multimap y)$. Practically, this is a proof of the following:

Corollary 3.2. *Given a frame* L, *the quintuple* $(L, \leqslant, \wedge, 1, \Rightarrow)$, *where* $\Rightarrow$ *is the exponential, is a lineale*.

We are now ready to define the family of categories $\mathbf{Dial}_L\mathbf{Set}$ [12]:

Definition 3.5. Let $(L, \leqslant, \circ, 1, \multimap)$ be a lineale. Then the objects of a category $\mathbf{Dial}_L\mathbf{Set}$ are triplets (X, r, A), where X and A are arbitrary sets and $r : X \times A \to L$ is function. Suppose that (X, r, A) and (Y, s, B) are two objects. Then a morphism between them is the pair (f, g), where $f : X \to Y$ and $g : B \to A$, such that

$$r(x, g(b)) \leqslant s(f(x), b), \quad \forall x \in X, \forall b \in B.$$

Note that any category $\mathbf{Dial}_L\mathbf{Set}$ is a symmetric monoidal closed category with involution and products and it is a categorical model of intuitionistic linear logic.

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[2] The expression *fully embeds* is an alternative to the more common term *the embedding is full*.

By following a line of arguments similar to those employed for Theorem 3.1, we can prove the following theorem:

Theorem 3.2. *Any category* **SET**(L) *embeds fully into a Dialectica category* **Dial**$_L$**Set**.

4. Discussion

In [9], the authors present an effort to "reconcile two different attempts to come to grips with the foundation of mathematics." One is mathematical logic and the other is category theory. The resulting theory is called categorical logic. Strictly speaking, categorical logic is an alternative presentation of logic.

In categorical logic categories are viewed as deductive systems. In particular, a categorical combinator $\phi : A \rightarrow B$ (i.e., a morphism from A to B) is the representation of a proof $A \vdash B$. In addition, the very existence of this combinator can be thought of as the "reason" why A entails B.

As we have already pointed out any category **Dial**$_L$**Set** is a symmetric monoidal closed category with involution and products and it is a presentation of intuitionistic linear logic. In addition, any category **Chu**(K) is a presentation of classical linear logic. It is also a fact that there are many objects of both categories which are the images of Goguen sets. Since both families of categories are presentations of linear logic, one may prove theorems in linear logic using, among others, fuzzy terms. Furthermore, the connectives of linear logic are naturally presented by endo-functors. Obviously, if we are working in a subcategory with objects the images of Goguen sets, we can define fuzzy connectives as endo-functors. Practically, this means that we can employ fuzzy formulas and prove theorems about them. In addition, we can combine fuzzy formulas with linear logic formulas and prove "mixed" theorems. Let us now demonstrate this idea with an example.

We consider the formula $A \& B \multimap A$. This formula can be proved using the intuitionistic linear sequent calculus as follows:

$$\frac{\dfrac{}{A \vdash A}}{\dfrac{A\&B \vdash A}{\vdash (A\&B) \multimap A}}$$

The proof above can be presented with a number of categorical combinators. This is almost trivial, once we have defined the various connectives of linear logic as endo-functors. For example, the connective & (pronounced *with*), which yields the sum of two terms, is defined as follows:

Definition 4.1. The *sum* of any two Dialectica spaces (X, r, A) and (Y, s, B) is the triplet $(X + Y, t, A \times B)$, where $t((x, y), (0, a)) = r(x, a)$ and $t((x, y), (1, b)) = s(y, b)$.

Here $X + Y = \{0\} \times X \cup \{1\} \times Y$ is the direct sum of the sets X and Y. Suppose now that $(S, s, \{\sigma\})$ and $(T, t, \{\tau\})$ are two Dialectica spaces representing the Goguen sets (S, σ) and (T, τ), respectively. Obviously, these two Dialectica spaces can be used in the above proof. This, in turn, means that we have a proof of a formula in linear logic using fuzzy subsets as terms.

Now, it is relatively easy to define a subcategory with objects the images of the objects of some Goguen category. In this subcategory, we can define the various connectives of fuzzy set theory as endo-functors. For example, here is the object part of such an endo-functor:

$$(S, q, \{\sigma\}) \vee (S, r, \{\tau\}) = (S, t, \rho),$$

where $t(s, \rho) = \max\{q(s, \sigma), r(s, \tau)\}$. By continuing our construction, we populate our subcategory with objects that are generated by applying the linear connectives to the objects of the initial subcategory. At the same time, one probably has to adjust the definition of the fuzzy connectives so that they produce meaningful results in all possible cases. In the end we get a structure where we can reason in the framework of fuzzy set theory and linear logic! This is indeed a very important perspective and needs to be explored in depth.

In the short history of linear logic, there have been some other interpretation of formulas and proofs. For example, formulas are viewed as computational processes and proof structures as distributed systems. This means that one can explore the use of fuzzy set theory in concurrency theory through the link described above. In addition, games can be viewed as formulas and strategies as proofs, which means that one can explore fuzzy game theory in a new setting. But these are areas of active research and so we do not plan to get into the details here.

5. Conclusions

We have shown that we can partially embed Goguen sets into a Chu category, while we can fully embed any category $\mathbf{SET}(L)$ into some Dialectica category $\mathbf{Dial}_L\mathbf{Set}$. Since, these Dialectica categories are models of intuitionistic linear logic, we have actually found a link between fuzzy set theory and linear logic. In addition, we have presented some ideas that may be used to introduce fuzzy set theory in mainstream computer science.

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