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YET ANOTHER FUZZY MODEL FOR LINEAR LOGIC

APOSTOLOS SYROPOULOS
Greek Molecular Computing Group
366, 28th October Str.
GR-671 00 Xanthi, GREECE
email: `asyropoulos@yahoo.com`

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The construction of a new categorical fuzzy model for linear logic is presented. The construction is based on a general poset-valued model. Since the resulting categories are not identical to existing categories of all fuzzy sets, we investigate the relationship between the two categories. We conclude with very brief comments regarding the usefulness of this work.

Keywords: Fuzzy sets, linear logic, categorical models.

1. Introduction

Linear logic[1] was devised by Jean-Yves Girard in the mid-1980's and ever since it became very popular among computer scientists. From its inception till today, many researchers have proposed various models of either classical or intuitionistic linear logic, both algebraic[1,2] and categorical[3,4,5]. Even recently, Girard himself proposed a method to reconcile continuity and logic, which is intrinsically discrete in nature, by means of a non-categorical model of linear logic[6]. On the other hand, this attempt is rather different from more recent efforts to define fuzzy models of linear logic[3,7,8], that is, models of linear logic where the building blocks are fuzzy sets or fuzzy structures, in general. However, one must not forget that these models are based on the ability of the so-called "two-sided" categorical models of linear logic to represent a big number of different mathematical structures.

Quite recently, Andrea Schalk and Valeria de Paive proposed a new method for the construction of poset-valued model of linear logic[9]. By using this method as a starting point, it is not difficult to construct a categorical fuzzy model of linear logic, which however is not general enough.

In this note, we give a brief overview of the method developed by Schalk and de Paiva. Then we build a new fuzzy model of linear logic based on this method. Next, this construction is compared with standard categories of all fuzzy sets. We conclude with some remarks concerning this construction and its usefulness.

2. A Poset-valued Model of Linear Logic

As it has been already mentioned, Schalk and de Paiva have proposed a new method for constructing models of linear logic. The resulting models are based on the category $\mathbf{Rel}$ of sets and relations between them. The general definition of a category that under certain conditions is either a model of classical or intuitionistic linear logic follows[9]:

Definition 2.1 Let F be an endofunctor on $\mathbf{Rel}$ and let P be a poset. The category of P_F-sets, denoted by $P_F\mathbf{Set}$, is defined as follows:

- An object $\alpha : FA \to P$ is a functional (i.e., a relation that we know it is a function).
- A morphism $\phi : (\alpha : FA \to P) \to (\beta : FB \to P)$ is defined by a relation $R \subseteq A \times B$ such that $x\ (FR)\ y$ implies $\alpha(x) \leq \beta(y)$.

It has been proved that if P is a *lineale* (or a symmetric monoidal poset) and F a monoidal functor, then $P_F\mathbf{Set}$ is a symmetric monoidal category [9] (essentially, such categories are models of intuitionistic linear logic). To get a model of intuitionistic linear logic, P has to be both a lineale and a complete lattice. Since the notion of a lineale is not widely know, we provide its definition[10]:

Definition 2.2 The quintuple $(L, \leq, \circ, 1, \multimap)$ is a lineale if:

- $(L, \leq)$ is poset,
- $\circ : L \times L \to L$ is an order-preserving multiplication, such that $(L, \circ, 1)$ is a symmetric monoidal structure (i.e., for all $a \in L$, $a \circ 1 = 1 \circ a = a$).
- if for any $a, b \in L$ exists a largest $x \in L$ such that $a \circ x \leq b$, then this element is denoted $a \multimap b$ and is called the pseudo-complement of a with respect to b.

Notice that the operator '$\circ$' is a logical conjunction operator, which is not necessarily idempotent. In addition, 1 is not necessarily the top element of L.

3. An I-valued Model of Linear Logic

If we want to construct a real-valued, poset-valued fuzzy model of linear logic, we need to choose a suitable endofunctor of $\mathbf{Rel}$ and prove that $I = [0,1]$ is a lineale. Let us start with the endofunctor. We have found that the best choice for the endofunctor is the identify functor $\mathrm{Id}_{\mathbf{Rel}}$. In addition, it is a fact that I is a complete lattice; so the following assertion is necessary in order to complete the first part of our construction:

Proposition 3.1 *The quintuple* $(I, \leq, \wedge, 1, \Rightarrow)$ *is a lineale.*

Proof. First notice that $(I, \leq)$, where $\leq$ is the usual ordering, is a poset. Next, $(I, \wedge, 1)$, where $a \wedge b = \min(a, b)$, is a symmetric monoidal structure. Now, for every $a, b \in I$, there is an element $a \Rightarrow b$ such that $c \leq a \Rightarrow b$ if and only if $c \wedge a \leq b$. This element is the exponential element of the Heyting algebra $(I, \vee, \wedge, 1, 0)$ and has the properties of the $\multimap$ operator. $\square$

So we have the two ingredients to build the category $I_{\mathrm{Id}}\mathbf{Set}$, which is a model for intuitionistic linear logic. This new category is surprisingly similar to the category of all fuzzy sets defined by Goguen[11] as follows:

Definition 3.3 **Fuzz** is the category of all fuzzy sets whose objects are pairs (S, σ), where S is a set and $\sigma : S \to I$ is a function. Given two objects (S, σ) and (T, τ) a morphism between these two objects is a function $f : S \to T$ such that $\sigma \leq f^{\leftarrow}(\tau)$.

Notice that given a lattice L and a function $f : X \to Y$, the preimage operator $f^{\leftarrow} : L^X \to L^Y$ is defined by $f^{\leftarrow}(b) = b \circ f$. Let us return to our construction. The next step is to see what is the relationship between $I_{\mathrm{Id}}\mathbf{Set}$ and **Fuzz** and to formalize it. Such a relationship is best described by a functor:

Definition 3.4 Functor $\mathcal{H} : \mathbf{Fuzz} \to I_{\mathrm{Id}}\mathbf{Set}$ is defined as follows:

- **Object part**: Let $\xi : A \to I$ be an object of **Fuzz**, then $\mathcal{H}(\xi)$ is the functional obtained from ξ.
- **Morphism part**: Let $\xi : A \to I$ and $\eta : B \to I$ be two objects of **Fuzz**; also let $f : A \to B$ be a morphism between them, then $\mathcal{H}(f) = \hat{f}$, where $\hat{f}$ is the graph of the function f.

First we note that the functor is representative because each object ξ of $I_{\mathrm{Id}}\mathbf{Set}$ is isomorphic to itself! Now consider two objects $\eta : A \to I$ and $\xi : B \to I$ and two parallel morphisms $f_1, f_2 : \eta \to \xi$ such that $f_1 \neq f_2$. Then this implies that $\mathcal{H}(f_1) \neq \mathcal{H}(f_2)$, which means that the functor is faithful. However, the functor is not full as not all relations are functionals. In other words,

Theorem 3.1 *Functor* $\mathcal{H} : \mathbf{Fuzz} \to I_{\mathrm{Id}}\mathbf{Set}$ *is representative and faithful.*

Now it is very easy to find a wide sub-category of $I_{\mathrm{Id}}\mathbf{Set}$, which is equivalent to $\mathbf{Fuzz}$. We shall call this sub-category $\mathbf{RelFuzz}$. The objects of $\mathbf{RelFuzz}$ are the objects of $I_{\mathrm{Id}}\mathbf{Set}$; and its morphisms are all those morphisms of $I_{\mathrm{Id}}\mathbf{Set}$ that are functionals. It is not hard to verify the following:

Theorem 3.2 *The (sub-)category* $\mathbf{RelFuzz}$ *is a categorical model for intuitionistic linear logic.*

4. Conclusions

We have presented the construction of a new category, which is a categorical model for intuitionistic linear logic. This new category is not equivalent to the standard category of all fuzzy sets and morphisms between them. However, a particular subcategory of the newly constructed category is indeed equivalent to the category of all fuzzy sets. So we have a category with objects fuzzy sets that is a model for intuitionistic linear logic. Since both linear logic and fuzzy sets have found many applications, it follows that by means of this new category, we can find applications of linear logic in fuzzy set theory and vice versa. This, in turn, is very interesting from a theoretical as well as from a practical point of view. For example, one can use fuzzy sets as terms in a linear deduction process.

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